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Conformal structure in the spectrum of an altered quantum Ising chain

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Abstract. The Ising model with an infinite line of defects is mapped onto a strip with two defect lines. The Hamiltonian spectrum is studied at the bulk critical point. Its exact diagonal form is found for an infinite number of sites. The spectrum of physical excitations contains an infinite number of primary fields, while the leading ground-state energy correction is independent of the defect strength. A novel algebraic structure interpolating between those belonging to periodic and free boundary conditions is signalled.

1. Introduction

In addition to scale invariance, a statistical model at its critical point also exhibits conformal invariance. In two dimensions conformal invariance plays a central role in constructing a systematic theory of critical correlations, first recognised by Belavin *et al* (1984) (for a review, see Cardy 1986a). The spectrum of the transfer matrix (or the Hamilton operator in the τ-continuum limit) is then classified into the irreducible representations of the Virasoro algebra, characterised by the highest-weight (primary) fields. Von Gehlen and Rittenberg showed, using finite-size scaling of the spectra of finite-width strips, how to identify the primary fields for specific models like the Z_3 Potts model (von Gehlen and Rittenberg 1986a, Rittenberg 1986) or the Ashkin-Teller model (von Gehlen and Rittenberg 1986b, 1987). Recently the investigation of the universality features of higher-order finite-size corrections was started, exploiting the knowledge of n-point correlation functions (Reinicke 1986).

The Ising model is known to be the simplest example of a conformally invariant field theory at its critical points (Belavin *et al* 1984). A detailed finite-size scaling study of its conformal properties based on the exactly known spectrum was done recently by Henkel (1987).

In this work, we shall study the influence of a defect line on the Ising model. Across this line the value of the coupling is κJ_c whereas for the rest of the lattice in the critical point its value is J_c. $\kappa = 1$ means no defect, while for $\kappa = 0$ a free surface is introduced into the system. Following Turban (1985) the infinite lattice is mapped onto a strip with two defect lines under the assumption that conformal invariance holds for the defected system too (figure 1). In the case of the so-called ladder-type defect the defects appear in the potential part of the Hamiltonian in well defined positions:

$$H = -\tfrac{1}{2}\sum_{i=1}^{N} \sigma_i^Z - \tfrac{1}{2}\sum_{i\neq N/4,3N/4} [\sigma_i^X\sigma_{i+1}^X + \kappa(\sigma_{N/4}^X\sigma_{N/4+1}^X + \sigma_{3N/4}^X\sigma_{3N/4+1}^X)]. \qquad (1.1)$$

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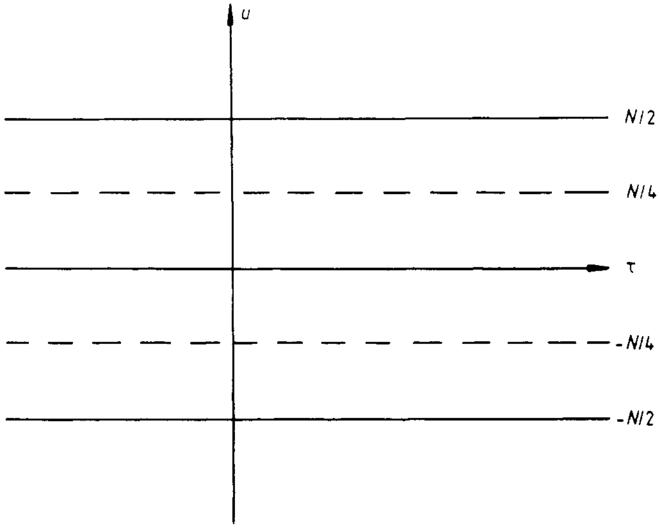

Figure 1. The finite-width strip with two parallel defects (broken lines) arising from conformal mapping of the infinite plane with function $w = (N/2\pi)\ln z$.

Although some critical exponents, $x(\kappa)$, vary continuously with κ, one still can derive the amplitude-exponents relation for the fundamental gap of H (Turban 1985):

$$A(\kappa) = 2\pi x(\kappa). \tag{1.2}$$

This relation has been checked to great accuracy for the two lowest excited levels by Guimarães and Drugowich de Felicio (1986) after performing a Jordan–Wigner (1928) transformation on (1.1). This yields the well known fermionic representation of (1.1) (see Lieb *et al* 1961):

$$\begin{aligned} H = \sum_{n=1}^{N} (C_n^\dagger C_n - \tfrac{1}{2}) - \tfrac{1}{2} \sum_{n=1}^{N-1} (C_n^\dagger - C_n)(C_{n+1}^\dagger + C_{n+1}) \\ + (\tfrac{1}{2} - Q)(C_N^\dagger - C_N)(C_1^\dagger + C_1) \\ + \tfrac{1}{2}(1-\kappa)[(C_{N/4}^\dagger - C_{N/4})(C_{N/4+1}^\dagger + C_{N/4+1}) \\ + (C_{3N/4}^\dagger - C_{3N/4})(C_{3N/4+1}^\dagger + C_{3N/4+1})] \end{aligned} \tag{1.3}$$

where $Q = 0$ denotes the energy sector (excitations are built of an even number of fermionic states), while $Q = 1$ stands for the spin sector with an odd number of filled fermionic levels.

Guimarães and Drugowich de Felicio (1986) also pointed out that the κ dependence of the exponents comes from the marginality of the defect energy operator (see also Brown 1982).

The exponents defined by the scaled gap values

$$(N/2\pi)G = F(\kappa) = x(\kappa) \tag{1.4}$$

characterise the correlation of spins at finite distance from the straight defect line in the two-dimensional plane. The most systematic study of the n-point functions with the n points aligned on the defect line was presented by Kadanoff (1981). His results are extremely helpful for the identification of all independent scaling fields appearing in the altered Ising model.

Our paper presents the following results concerning the conformal properties of the Ising model in the $0 \leq \kappa \leq 1$ interval.

(i) We find, in the $N \to \infty$ limit, the exact diagonal form of equation (1.3)

$$H = \frac{2\pi}{N} \sum_{r=0}^{\infty} [(\tfrac{1}{2} - \Delta + r)\eta^{\dagger}_{r(1)}\eta_{r(1)} + (\tfrac{1}{2} + \Delta + r)\eta^{\dagger}_{r(2)}\eta_{r(2)}] + \text{constant} \tag{1.5}$$

where Δ can take all values between 0 ($\kappa = 0$) and $\frac{1}{2}$ ($\kappa = 1$). We shall show how to parametrise Δ as a function of κ (see § 4).

(ii) In the energy sector (which is generated by the identity and the energy operators) no κ dependence is observed. The leading $O(1/N)$ correction to the ground-state energy stays fixed at its value computed (Affleck 1986, Blöte *et al* 1986) for the $\kappa = 1$ case from the value of the central charge, $c = \frac{1}{2}$.

(iii) In the spin sector infinitely many exponents vary continuously with κ. While for $\kappa = 0, 1$ the spectrum consists of just two conformal towers, we find in the whole $0 < \kappa < 1$ interval infinitely many new primary fields. Denoting the scaling dimension of the kth primary field by x_k, we find

$$(2x_{k+1})^{1/2} = 2k - (2x_k)^{1/2}. \tag{1.6}$$

This relation is shown to be a direct consequence of the diagonal form of the Hamiltonian.

(iv) The whole spectrum can be classified into towers starting with the primary fields of equations (1.6). The degeneracy of the secondary levels was found to follow from simple generating functions.

(v) Results listed under (ii) and (iii) exactly reproduce the conclusions of the Euclidean analysis by Kadanoff (1981). This fact represents the complete verification of the correctness of Turban's mapping, which is not *a priori* self-evident, since in the case of free boundary conditions only real analytic mappings are allowed (Cardy 1984).

The infinite number of primary fields, the degeneracy pattern and the fixed coefficient in front of the $1/N$ correction to the ground-state energy signal the appearance of an original conformal algebraic structure in the defected Ising system.

The paper is organised as follows. In § 2, to make the presentation self-contained, we review the results of Kadanoff (1981) and reformulate them to be of maximal use for the identification of the conformal structures in the spectrum. The spectrum itself is analysed in the following sections. In § 3, we perform analytic perturbative calculations around the exactly known solutions for $\kappa = 1$ and 0.

Section 4 is the heart of the paper. The spectrum of (1.3) is found numerically for any value of κ. In the limit $N \to \infty$, the exact diagonal form (see equation (1.5)) of the Hamiltonian is obtained. The primary fields and their towers are identified.

Our conclusions are given in § 5.

2. Primary fields in presence of a straight-line defect

In this section the results of Kadanoff (1981) will be described in order to make our discussion self-contained. We also introduce the notations of the present paper.

Consider the composite field

$$O_N = \prod_{j=1}^{N} D_{\gamma_j}(y_j) \tag{2.1}$$

with the $D_\gamma(y)$ variables ordered along the y axis in the plane. The index refers to the type of the field as follows (Kadanoff 1981)

$$\gamma = \begin{cases} 0 & \text{unit operator} \\ \frac{1}{2} & \text{order parameter} \\ -\frac{1}{2} & \text{disorder parameter} \\ 1 & \text{fermion} \\ -1 & \text{antifermion.} \end{cases} \tag{2.2}$$

A 'quantum number' Γ_N is associated with this set of operators, defined recursively

$$\Gamma_{j+1} = \Gamma_j + (-1)^{2\Gamma_j}\gamma_{j+1} \qquad \Gamma_1 = \gamma_1. \tag{2.3}$$

It was shown by Kadanoff and Ceva (1971) that the expectation value of O_N is different from zero only when $\Gamma_N = 0$ (the 'neutrality condition'). Then at the critical point a Coulomb gas type expression was found:

$$\langle O_N \rangle = e^\Lambda \qquad \Lambda = \sum_{j>i} q_i q_j \ln(y_j - y_i). \tag{2.4}$$

Clearly the anomalous dimensions of the fields are determined by their charges q_i.

In the perfect infinite system the charge of a field at site y_j is simply given by

$$q_j = (-1)^{2\Gamma_{j-1}}\gamma_j \tag{2.5}$$

yielding the following scaling dimensions for the 'elementary fields' ($x = \frac{1}{2}q^2$):

$$x_{1/2} = x_{-1/2} = \tfrac{1}{8} \qquad x_1 = x_{-1} = \tfrac{1}{2}. \tag{2.6}$$

The composite fields appear in the short distance expansion of the product of two elementary fields possessing the following fusion rules:

$$\begin{aligned} &D_\alpha(y+0)D_\beta(y-0) \sim D_\gamma(y) \\ &\gamma = \alpha + (-1)^{2\alpha}\beta \\ &q_\gamma = q_\alpha + (-1)^{2\alpha}q_\beta. \end{aligned} \tag{2.7}$$

According to (2.7) n fermions (antifermions) form a composite field of charge n ($-n$). The fusion of these fields with an order (disorder) variable (approaching the composite from the direction of smaller y values) builds up variables with half-integer charge.

The scaling dimensions of the operators created in the short distance expansion are again calculated by the formula $x = \frac{1}{2}\gamma^2$. The integer charged fields are seen to belong either to the conformal tower of the unit and the energy operators (when x is integer) or to that of the fermionic fields (x is half integer). The half-integer charges have dimensions $x = \frac{1}{8} + \{[\frac{1}{2}(2n+1)]^2\frac{1}{2} - \frac{1}{8}\}$ which belong therefore to the order or disorder towers. This means that no new primary field occurs among the composite fields when the system is perfect.

Now, along the line on which the multi-point correlation functions are evaluated, the nearest-neighbour coupling is enhanced by a factor $\bar{\kappa}$. We shall concentrate on the so-called ladder-type defects, where the variation of $\bar{\kappa}$ between 0 and 1 interpolates between the perfect infinite system and two half planes with free boundary conditions. The conformal characterisation of both these systems is well known (Cardy 1986b). The interest of the present study lies in understanding the interpolating conformal structures.

The expressions (2.3), (2.4) and (2.7) remain valid under this modification in unchanged form. Equation (2.5) is replaced by a 'renormalised' expression, in which in place of γ some function of $\bar{\kappa}$ appears (Kadanoff 1981):

$$g(\gamma, \bar{\kappa}) = \begin{cases} \gamma + (1/\pi)\sin^{-1}(\tanh 2\bar{\kappa}) & \gamma = \frac{1}{2}(2n+1) \\ \gamma & \gamma = n. \end{cases} \tag{2.8}$$

The scaling dimension of the fields built up from those characterised by integer γ are therefore unchanged. The neutrality condition $\Gamma_N = 0$ which reflects the actual value of the central charge (Dotsenko and Fateev 1984) implies a c value which is independent of $\bar{\kappa}$ as well.

The additive renormalisation of all the charges in the order (disorder) parameter sectors actually originates from the 'non-renormalisation' of the integer charges. The spacing of the continuously changing scaling dimensions $x_\gamma = \frac{1}{2}g^2(\gamma, \bar{\kappa})$ is not quantised any more. This fact signals the occurrence of infinitely many new primary fields in the system with defects.

As the scaling dimensions of these fields were given for $\bar{\kappa} = 1$ as $x - \frac{1}{8} = n(n+1)/2$ ($= 1, 3, 6, 10, \ldots$), the levels of the perfect system, in the neighbourhood of which the defect gives rise to new primary fields, are uniquely foreseen.

The above observations make it explicit that the Ising model with line defects is by no means minimal (the number of primary fields is not finite). Still, the dimensions of the new primary fields are not arbitrary; they can all be parametrised with the help of a single one, say $x_\mu(\bar{\kappa})$, the continuously changing dimension of the disorder variable. (As usual, μ refers to the disorder and σ to the order parameter.)

The non-renormalisation of the fermion charges relates $x_\mu(\bar{\kappa})$ to $x_\sigma(\bar{\kappa})$. As the defect line breaks self-duality, they are no longer equal, but by equation (2.7) the product of an order and a disorder field gives a fermion (antifermion) as before. Then (2.8) and (2.7) require

$$g(\tfrac{1}{2}, \bar{\kappa}) + g(-\tfrac{1}{2}, \bar{\kappa}) = (2x_\sigma(\bar{\kappa}))^{1/2} + (2x_\mu(\bar{\kappa}))^{1/2} = 1 \tag{2.9}$$

a relationship first recognised by Brown (1982). For the notation of the dimensions of the family of new primary fields we shall use $x_\sigma(n, \bar{\kappa})$ and $x_\mu(n, \bar{\kappa})$ with $x_\sigma(0, \bar{\kappa}) = x_\sigma(\bar{\kappa})$, $x_\mu(0, \bar{\kappa}) = x_\mu(\bar{\kappa})$. They are given as

$$\begin{aligned} (2x_\sigma(n, \bar{\kappa}))^{1/2} &= (2x_\sigma(\bar{\kappa}))^{1/2} + n = n + 1 - (2x_\mu(\bar{\kappa}))^{1/2} \\ (2x_\mu(n, \bar{\kappa}))^{1/2} &= (2x_\mu(\bar{\kappa}))^{1/2} + n. \end{aligned} \tag{2.10}$$

This is nothing else than equation (1.6), where the series x_μ belongs to the odd and the series x_σ to the even values of k.

The considerations based on (2.4) do not suffice, however, to ascertain whether the new independent fields do possess their own conformal towers. This question can be answered most directly by finding the spectrum of the corresponding Hamilton operator on a finite-width strip. The degeneracy structure of the levels provides information concerning the way the conformal symmetry is realised by the Ising model in the interval $0 < \kappa < 1$.

3. Results from a perturbative analysis

In this section we shall determine the excitation spectrum of (1.3) perturbatively. We find features which prove to be present for any value of κ. Therefore readers not interested in analytic formulae might skip this section and proceed directly to § 4.

The eigenvalues of (1.3) are obtained with the help of matrices appearing when (1.3) is reshuffled into the form

$$H = -\sum_{n,m} [C_n^\dagger A_{nm} C_m + \tfrac{1}{2}(C_n^\dagger B_{nm} C_m^\dagger + \text{HC})] \tag{3.1}$$

(Lieb *et al* 1961). A is a symmetric and B an antisymmetric real $N \times N$ matrix. The eigenmodes Φ, ψ and eigenvalues Λ are determined by the set of equations

$$\begin{aligned} \Phi_n(A-B)_{nm} &= \Lambda\psi_m \\ \psi_n(A+B)_{nm} &= \Lambda\Phi_m \end{aligned} \tag{3.2}$$

which is equivalent to finding the eigenvalues of the symmetric matrix

$$M = (A-B)(A+B). \tag{3.3}$$

The perturbative analysis of this eigenvalue problem parallels that performed by Brown (1982) in the Euclidean formulation for $\kappa \simeq 1$. In particular, we are able to make explicit the way marginal operators produce continuously moving exponents in the Hamiltonian formulation of the theory.

3.1. $\kappa \simeq 0$

In this case one separates terms proportional to κ in (1.3) as perturbations to the system consisting of two free strips of width $N/2$. The perturbation couples these strips through nearest-neighbour couplings across the defect lines:

$$\begin{aligned} [\Delta(A-B)]_{mn} &= \kappa[\delta_{n,N/4+1}\delta_{m,N/4} + \delta_{n,3N/4+1}\delta_{m,3N/4}] \\ &= [\Delta(A+B)]^{\mathrm{T}}_{mn}. \end{aligned} \tag{3.4}$$

The physical excitations at $\kappa = 0$ are mere juxtapositions of the excitations of two independent strips. The unperturbed levels and eigenfunctions are therefore already computed (Lieb *et al* 1961, Burkhardt and Guim 1985).

The application of the perturbation theory to leading order in κ does not lift the double degeneracy of the fermionic levels in the $Q = 0$ sector. Since the spectrum of a single strip is given by the representations of one Virasoro algebra, the spectrum of the energy sector of H (equation (1.1)) is given by the direct product of two representations of the Virasoro algebra. In particular, the expression for the ground-state energy is

$$E_0 = 1 - \operatorname{cosec} \pi/2(N+1) \tag{3.5}$$

(see Burkhardt and Guim 1985). This implies also that the leading $\mathrm{O}(1/N)$ correction to the bulk energy density is given by

$$2\frac{\pi c}{24}\frac{1}{N/2} = \frac{\pi c}{6}\frac{1}{N} \tag{3.6}$$

which exactly coincides with the correction of one periodic chain of length N (Affleck 1986, Blöte *et al* 1986).

The perturbation (3.4) in the $Q = 1$ spin sector leads to the following corrected values of Λ:

$$\begin{aligned} \Lambda_{u,\mp} &= 2\sin\tfrac{1}{2}\delta_u \mp (4\kappa/N)\cos\tfrac{1}{2}\delta_u \sin\tfrac{1}{2}N\delta_u \\ &= (\pi/N)[2u+1\mp(-1)^u(4\kappa/\pi)] + \mathrm{O}(1/N^2) \end{aligned} \tag{3.7}$$

where

$$\delta_u = \frac{2u+1}{N+1}\pi \qquad u = 0, 1, \ldots, \tfrac{1}{2}N - 1.$$

According to (3.7), the fermionic levels in the spin sector split symmetrically around their $\kappa = 0$ positions and form two infinite towers starting with the highest weights:

$$\Lambda_1^{(1)} = \frac{2\pi}{N}\left(\frac{1}{2} - \frac{2\kappa}{\pi}\right) \qquad \Lambda_1^{(2)} = \frac{2\pi}{N}\left(\frac{1}{2} + \frac{2\kappa}{\pi}\right). \tag{3.8}$$

The lowest scaled gap can be calculated with the help of (3.5) and (3.7) from the formula

$$F_{\text{gap}} = \frac{N}{2\pi}(E_0(Q=1) - E_0) + \frac{N}{2\pi}\Lambda_1^{(1)}$$

where $E_0(Q=1) = -\frac{1}{2}\Sigma_{u,\pm}\Lambda_{u,\pm}$. This expression, when expanded to $O(N^0)$, gives

$$F_{\text{gap}} = \tfrac{1}{2} - 2\kappa/\pi \tag{3.9}$$

which by the amplitude exponent relation (1.4) agrees with the Hamiltonian expression of $x = \eta/2$ (Peschel and Schotte 1984) expanded to $O(\kappa)$.

The general analysis of the perturbative finite-size corrections to the scaled gap from local operators with scaling dimension x yields an N dependence $\sim N^{1-x}$ (Cardy 1986b, Rittenberg 1986). The leading effect of a marginal ($x = 1$) perturbation is therefore to shift the gap amplitude by a quantity proportional to κ. The nearest-neighbour energy operator is just of this type. We stress that our formulae indicate $O(1/N^2)$ next-to-leading finite-size corrections to the eigenvalues and no $O(N \log N)^{-1}$-type corrections seem to appear.

3.2. $\kappa \simeq 1$

The perturbation part of the Hamiltonian is now proportional to $(1-\kappa)$ and one uses in the standard perturbative calculations the periodic $\kappa = 1$ wavefunctions and energy levels of Burkhardt and Guim (1985).

In the $Q = 0$ sector the corrected levels are

$$\Lambda_u = [1 - (1-\kappa)/N]\Lambda_u^{(0)} \tag{3.10}$$

where $\Lambda_u^{(0)}$ stands for the unperturbed eigenvalues. The ground-state energy $E_0 = -\frac{1}{2}\Sigma_u \Lambda_u$ then receives the same multiplicative modification and is given (Burkhardt and Guim 1985) by

$$E_0 = [1 - (1-\kappa)/N]/\sin(\pi/2N) \tag{3.11}$$

which for $N \to \infty$ yields the surface energy $2(1-\kappa)/N$, but leaves the $O(1/N)$ correction term unaffected. This implies that the leading ground-state energy correction, governed at $\kappa = 1$ by the central charge $c = \frac{1}{2}$, is unchanged.

In the $Q = 1$ sector we give only the $O(\kappa)$ expressions for the lowest eigenvalue Λ_0 and the contribution to the zero-point energy $E_0(Q=1) = -\frac{1}{2}\Sigma_u \Lambda_u(Q=1)$ which determine the lowest gap amplitude:

$$\begin{aligned} &\Lambda_0 = 2(1-\kappa)/N \\ &E_0(Q=1) = -[1-(1-\kappa)/N]\cot(\pi/2N) - (1-\kappa)/N + O(1/N^2). \end{aligned} \tag{3.12}$$

Then the scaled gap is given as

$$F_{\text{gap}} = (N/2\pi)(E_0(Q=1) + \Lambda_0 - E_0) = \tfrac{1}{8} + (1-\kappa)/2\pi \tag{3.13}$$

which again agrees with the Hamiltonian result of Peschel and Schotte (1984), linearised in $1-\kappa$.

Although the exponents move continuously with κ, we claim at least a restricted universality. Following Henkel (1987) we have studied the quantum XY chain with defects:

$$
\begin{aligned}
H = -\frac{1}{2\gamma}\sum_{i=1}^{N}\sigma_i^Z - \frac{1}{2\gamma}\sum_{i\neq N/4,3N/4}[\tfrac{1}{2}(1+\gamma)\sigma_i^X\sigma_{i+1}^X+\tfrac{1}{2}(1-\gamma)\sigma_i^Y\sigma_{i+1}^Y] \\
-(\kappa/2\gamma)[\tfrac{1}{2}(1+\gamma)(\sigma_{N/4}^X\sigma_{N/4+1}^X+\sigma_{3N/4}^X\sigma_{3N/4+1}^X) \\
+\tfrac{1}{2}(1-\gamma)(\sigma_{N/4}^Y\sigma_{N/4+1}^Y+\sigma_{3N/4}^Y\sigma_{3N/4+1}^Y)]
\end{aligned}
\tag{3.14}
$$

which for $\gamma\neq 0$ should fall into the same universality class as the defected Ising model, equation (1.1). From the same perturbative analysis, we indeed obtain the same conclusions as for the Ising model above.

In the next section we show that the presence of two fermionic towers (equation (3.7)) generalises for arbitrary value of κ and comprises all the information about the new conformal structure appearing in the interval $0<\kappa<1$.

4. Fermionic representation of the scaling fields

The numerical diagonalisation of (1.3) leads in both sectors to results of appealing simplicity. This simplicity allows us to guess the exact form of the diagonalised Hamiltonian for $N\to\infty$. As we shall show in this section, from this Hamiltonian all the results listed in the introduction can be derived.

The single fermionic spectrum of the $Q=0$ sector emerging from the numerical diagonalisation of the M matrix of equation (3.3) proves to be independent of κ, i.e. the diagonalised Hamiltonian is of the same form as found by Lieb *et al* (1961). The conformal features of the energy sector do not differ from the $\kappa=1$ perfect Ising case.

The results in the $Q=1$ sector are best summarised by giving the diagonal form of the Hamiltonian for $N\to\infty$

$$\frac{N}{2\pi}H=\sum_{r=0}^{\infty}(\Lambda_r^{(1)}\eta_{r(1)}^{+}\eta_{r(1)}+\Lambda_r^{(2)}\eta_{r(2)}^{+}\eta_{r(2)})+\text{constant} \tag{4.1}$$

where the two sequences of scaled (!) eigenvalues were found to seven digits to allow the following parametrisation:

$$
\begin{aligned}
\Lambda_r^{(1)} &= \tfrac{1}{2}-\Delta(\kappa)+r+O(1/N) \\
\Lambda_r^{(2)} &= \tfrac{1}{2}+\Delta(\kappa)+r+O(1/N).
\end{aligned}
\tag{4.2}
$$

Here, $\Delta(\kappa)$ is a smooth function of the defect strength. Its analytic functional form will be found from the requirement of reproducing the results of the Euclidean analysis. The relation (4.2) has been verified up to $r=12$ in each tower by applying the Bulirsch-Stoer (1964) convergence improving algorithm to the spectrum of chains containing up to 210 sites.

We proceed now to finding all primary fields. It turns out that we have infinitely many of them. In the spin sector all states are built from an odd number of fermionic excitations. Since $\Delta(\kappa)$ changes continuously with κ and because by equation (3.20) $\Lambda_r^{(1)}+\Lambda_{r'}^{(2)}=r+r'+1$ the only candidates for primary fields are those states which arise by exciting the vacuum exclusively by one set of fermionic modes. The scaled gap amplitudes (F) defined by multiplying the eigenvalues of (1.3) by $N/2\pi$ are given by

$$F_p^{(1)} = \sum_{r=0}^{2p} \Lambda_r^{(1)} + \Delta E$$
$$F_p^{(2)} = \sum_{r=0}^{2p} \Lambda_r^{(2)} + \Delta E \tag{4.3}$$

where ΔE is the difference in the zero-point energies of the $Q=0$ and $Q=1$ sectors. The states where some of the lower fermionic levels are empty or, in addition, pairs of $\Lambda_r^{(1)}$ and $\Lambda_r^{(2)}$ modes are filled, will be secondaries of one of the primary fields given by equation (4.3).

In order to reproduce results of the Kadanoff analysis (§ 2), we require (4.3) to take the form

$$F_p^{(i)} = \tfrac{1}{2} q_p^{(i)2} = \tfrac{1}{2}(2p+1+\alpha^{(i)})^2 \tag{4.4}$$

which determines $\Lambda_0^{(i)}$ and ΔE when the equidistant spacing of the single fermion levels is taken into account in (4.3):

$$2\alpha^{(i)} = 2\Lambda_0^{(i)} - 1 = (-1)^i 2\Delta$$
$$\Delta E = \tfrac{1}{2}\alpha^{(i)2} = \tfrac{1}{2}\Delta^2. \tag{4.5}$$

Comparing (4.5) with (2.10) one is led to the conclusion that the series $F_p^{(1)}$ coincides with every *second* term of the series of primary fields starting with $x_\sigma(\kappa)$ ($n = 0, 2, 4, \ldots$) found in the Kadanoff analysis provided

$$\tfrac{1}{2} - \Lambda_0^{(1)} = (2x_\mu(\kappa))^{1/2} = 1 - (2/\pi)\tan^{-1}(1/\kappa)$$
$$\Delta E = x_\mu(\kappa). \tag{4.6}$$

Then the primary dimensions $F_p^{(2)}$ are identified with the $n = 1, 3, 5, \ldots$ terms of the series starting with $x_\mu(\kappa)$ in (2.10). The dimensions not appearing in the spin sector occur by a simple duality argument in the disorder-excitation (kink) channel.

This means that the verification of Turban's mapping for the full spectrum is reduced to the check of the agreement of (4.6) with the data obtained numerically. This is done in table 1, where we show values of $\Lambda_0^{(1)}$ and ΔE as calculated from (4.6) for various values of κ. In the next two columns, denoted by $\tilde{\Lambda}_0$ and $\tilde{\Delta} E$, we give the values for these quantities as determined by extrapolating (Bulirsch and Stoer 1964)

Table 1. Check of the conformal invariance of the defected Ising model. The quantities of columns two and three are calculated from equation (4.6), while those of columns four and five are extracted from numerical solution of (1.3).

κ	$\Lambda_0^{(1)}$	ΔE	$\tilde{\Lambda}_0$	$\tilde{\Delta} E$
1.0	0	$\frac{1}{8}$	0	$\frac{1}{8}$
0.9	0.033 4754	0.108 8225	0.033 4754	0.108 8226
0.7	0.111 1997	0.075 5828	0.111 1998	0.075 5827
0.5	0.204 8327	0.043 5618	0.204 8328	0.043 5619
0.3	0.314 4528	0.017 2139	0.314 4528	0.017 2139
0.1	0.436 5489	0.002 0130	0.436 549	0.002 0130
0.0	$\frac{1}{2}$	0	$\frac{1}{2}$	0

the finite lattice data from chains up to 210 sites. The agreement of our data and equation (4.6) holds at least to six digits.

Therefore the $N \to \infty$ limit of the diagonalised Hamiltonian is parametrised analytically by

$$\Delta(\kappa) = 1 - (2/\pi)\ \tan^{-1}(1/\kappa). \tag{4.7}$$

Having established that the framework of conformal invariance is fully valid in the altered Ising model, we now display the Hamiltonian spectrum. In figure 2 we show the physical states in the spin sector for several values of κ up to levels $p = 2$ (e.g. equation (4.3)). The figure indicates a smooth interpolation between the two known extremes ($\kappa = 0, 1$). The dotted lines stand for levels belonging to primary fields while the full lines are secondary levels. The degeneracy structure provides further information about the nature of the symmetry present in the defected system for $0 < \kappa < 1$. In table 2, the classification of the first 294 lowest-lying spin excitations is given into the $p^{(1)} \leqslant 3$ and $p^{(2)} \leqslant 2$ conformal towers. In the limit $\kappa \to 1$ these towers collapse into the single tower of the periodic defectless Ising model.

The degeneracy $d_\kappa(N)$ of each level in a specific tower follows the same pattern represented by the generating function

$$\sum_{n=0}^{\infty} d_\kappa(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \tag{4.8}$$

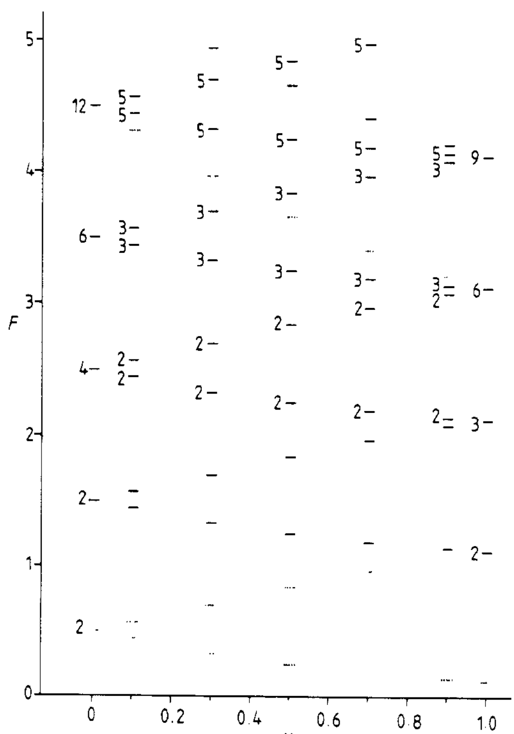

Figure 2. Low-lying spectrum in the spin sector as a function of κ. Dotted levels denote primary states and full lines their secondaries. The numbers give the degeneracy of each state.

Table 2. Degeneracies of the levels of the perfect ($\kappa = 1$) and the defected chain. The levels at $\kappa \neq 1$ arise from splitting of that level of the $\kappa = 1$ system which appears in the same row.

Perfect Ising model $\kappa = 1$		Defected Ising model $0 < \kappa < 1$					
Level (N)	$d(N)$	Primary field expected	$d_\kappa(N)$				
0	1	yes	1				
1	2	yes	1	1			
2	3	no	2	1			
3	6	yes	3	2	1		
4	9	no	5	3	1		
5	14	no	7	5	2		
6	22	yes	11	7	3	1	
7	32	no	15	11	5	1	
8	46	no	22	15	7	2	
9	66	no	30	22	11	3	
10	93	yes	42	30	15	5	1

This pattern occurs when states belonging to a given level of a tower are created by unconstrained application of some step operators 1_{-n} (to be constructed from fields $\eta_{r(1)}, \eta_{r(2)}$) to the highest-weight states $|i, p^{(i)}\rangle$. It is to be seen how these operators can be related to the generators of some symmetry describing the simple spectrum expressed by (4.1), (4.2) and (4.7).

5. Conclusions and prospects

We have studied the Ising model with a straight-line defect at criticality. In the limit $N \to \infty$, we found the exact spectrum of the Hamiltonian (1.3) as given by equations (4.1) and (4.2). This result is of great help for a detailed analysis of the spectrum.

We have been able to verify completely the validity of conformal invariance for the whole spectrum. In addition to that, we have identified the infinitely many new primary fields which emerge if κ is varied and our results agree with the Euclidean analysis of Kadanoff. The subleading finite-size corrections to the energy levels in both sectors are of the order $1/N^2$ and we stress the absence of logarithmic factors $O(1/N \log N))$ although we have a marginal operator in our model.

Moreover, we have established the complete degeneracy pattern for arbitrary values of κ and found an interesting relationship of the perfect periodic Ising Hamiltonian to the level degeneracies, which is expressed clearly in table 2. On the other hand, our understanding of the defected Ising chain is not complete. A better insight should be gained along the following lines of research.

(i) It is useful to study the Ising system with a half-infinite defect line, which is equivalent through conformal mapping to a Hamiltonian containing a single modified bond. A preliminary investigation revealed a structure similar to equation (1.5). This hints at the presence of a unique algebraic structure in both problems.

(ii) The exact diagonal form (1.5) of the Hamiltonian should be derived analytically.

(iii) The construction of the spectrum-generating algebra built up from the fields is in progress.

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Note added in proof. In recent papers (Henkel and Patkós 1986, 1987) the problems (i) and (iii) could be solved, resulting in the construction of a U(1) Kac–Moody–Virasoro algebra with $c=1$ generating the spectrum of the Hamiltonian with either one or two defects.

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